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2005 J. Phys. A: Math. Gen. 38 6939

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Group approach to the quantization of the Pöschl–Teller dynamics*

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Received 5 April 2005, in final form 15 June 2005

Published 20 July 2005

Online at stacks.iop.org/JPhysA/38/6939

Abstract

The quantum dynamics of a particle in the modified Pöschl–Teller potential is derived from the group $SL(2, \mathbb{R})$ by applying a group approach to quantization (GAQ). The explicit form of the Hamiltonian as well as the ladder operators is found in the enveloping algebra of this basic symmetry group. The present algorithm provides a physical realization of the non-unitary, finite-dimensional, irreducible representations of the $SL(2, \mathbb{R})$ group. The non-unitarity manifests itself in that only half of the states are normalizable, in contrast with the representations of $SU(2)$ where all the states are physical.

PACS number: 03.65.Fd

1. Introduction

Symmetry has proven very useful in quantum mechanics as a powerful tool to construct explicitly the eigenstates and eigenvalues of a given symmetrical Hamiltonian. Since the pioneering work of Wigner [1] many papers have been devoted to the analysis of solvable quantum systems through their ‘dynamical symmetries’ or ‘spectrum-generating algebras’ [2]. In particular, the Pöschl–Teller and Morse potentials, binding molecular systems, have been soundly studied along these lines [3–6] (see also [7] for a recent and more detailed bibliography). But symmetry can be taken beyond this ability and constitutes the fundamentals for physical systems in such a way that any referent to them, that is, spacetime, classical solution manifold, wavefunctions, operators, scalar product, etc, can be explicitly derived in a natural manner from a particular Lie group. This viewpoint has been demonstrated in many finite- and

* Work partially supported by the DGICYT and Fundación Séneca.

infinite-dimensional cases by applying a group approach to quantization (GAQ) developed since the original paper [8], where the quantum free Galilean particle and the harmonic oscillator were derived. Then, this algorithm has been applied to less elementary groups such as those associated with relativistic particles, in particular the relativistic harmonic oscillator [9–11], field theories in curved spacetimes, nonlinear σ -models, the Virasoro group and others concerning conformal symmetry and quantum gravity (see, for instance, [12–14]).

The modified Pöschl–Teller potential (MPT), however, has a special attraction in spite of its simplicity, because it seems not to be primarily associated with a particular symmetry, but rather with a phenomenological force, and it is less integrable than other more involved physical problems. In specific terms, the classical Hamiltonian does not close a Poisson subalgebra with the coordinate and the momentum. This system prompts us to search for an alternative finite-dimensional Poisson subalgebra in the free algebra generated by $\langle H, x, p \rangle$, a procedure which would be of a wide usage since these generators generally fail in closing a subalgebra in many physical systems. In fact, it is possible to find two classical functions, \mathcal{X} and \mathcal{P} , that close with the classical Hamiltonian H a ‘Lie algebra’ having the structure constants replaced with functions of the energy, a breakdown of the Lie structure similar to that found in the hydrogen atom when trying to close the dynamical symmetry generated by the angular momentum and the Runge–Lenz vector [15, 16], which turns out to be $SO(4)$, $E(3)$ or $SO(3, 1)$ depending on whether the fixed energy is negative, null or positive. Unlike the hydrogen atom, here the energy is not central in these Lie algebras, and the symmetry proves to be $SO(2, 1)$ (or $SU(1, 1) \approx SL(2, \mathbb{R})$) although for bound states (negative energy) a complex prolongation of this algebra can be confused with $SU(2)$ at the classical level. In fact, since the Lie algebras of these groups have the same complex form, a complex prolongation from one part of the spectrum to the other can be easily performed.

We shall proceed by taking the square root of H and considering the set $\langle \mathcal{E} \approx \sqrt{H}, \mathcal{X}, \mathcal{P} \rangle$, which closes a true Lie algebra, $SO(2, 1)$, as the starting point for the GAQ. From an algebraic group law for $SO(2, 1)$ we derive the irreducible representations of the group as well as the explicit expression of all operators in the (enveloping) algebra and, in particular, the operator $\mathcal{E}^2 \approx H$. This operator will result in $\hat{p}^2 - D/\cosh^2(\alpha x)$, i.e. the quantum operator representing the original Hamiltonian associated with the potential $V(x) = -D/\cosh^2(\alpha x)$, D being the absolute value of the potential depth and α an indicative of its width.

A remarkable feature appearing in the present process is that the eigenstates of the MPT Hamiltonian with negative energy, i.e. the bound states, are formally obtained from the wavefunctions in the discrete series of the $SL(2, \mathbb{R})$ unitary irreducible representations (the wavefunctions for a model of a relativistic harmonic oscillator) with negative Bargmann index $k = -q < 0$. This can be seen to correspond to a non-unitary, finite-dimensional, representation of $SL(2, \mathbb{R})$ [17] (see [18] for a different discussion, and [19] for an explicit realization of matrix elements in terms of hypergeometric functions). The non-unitarity of the representation reveals itself in the fact that not all the wavefunctions are normalizable, and therefore the physical Hilbert space is smaller. In fact, from the $2q + 1$ states of the representation, Ψ_n^q , $n = 0, \dots, 2q$, only $[q] + 1$ are normalizable (where $[q]$ stands for the greatest integer properly smaller than q). If q is an integer, there are only q states, from $n = 0, \dots, q - 1$, the state with $n = q$ (which correspond to zero energy) being not normalizable. If q is half-integer, there are $q + \frac{1}{2}$ states, from $n = 0, \dots, q - \frac{1}{2}$. Going to the universal covering group of $SL(2, \mathbb{R})$ real values of q are also allowed.

This behaviour is very different from that of $SU(2)$, where the representations are $2j + 1$ dimensional, with j integer or half-integer, but all states are normalizable since the representations are unitary. This shows that the correct symmetry for bound states is not $SU(2)$, as is normally claimed in the literature (see, for instance, [5, 7]), but rather the

finite-dimensional representations of $SL(2, \mathbb{R})$. Note that in the quantum description of the MPT system in terms of $SU(2)$ non-half-integer values of q would be forbidden.

We remind the reader that for a non-compact, simple group G , such as $SL(2, R)$, a finite-dimensional representation ρ is not unitary because, otherwise, the group $\{\rho(g)\}$ would be a subgroup of $U(n)$ ($n = \dim \rho$) and, therefore, $\{\rho(g)\}$ and G themselves compact [20].

This paper is organized as follows. In section 2 the classical dynamics in the MPT potential is presented aiming at finding the relevant symmetry that will be quantized in abstract terms in the framework of the GAQ. Section 3 is devoted to a very brief report on the GAQ with the example of the relativistic harmonic oscillator. The quantum dynamics associated with the MPT interaction, as well as the corresponding Hamiltonian and ladder operators, is derived in section 4 and, finally, some conclusions are drawn in section 5.

2. Classical theory and Poisson symmetry

Even though the GAQ is primarily intended to achieve quantum systems without the previous step of solving the classical counterpart, the classical theory can help us in finding the relevant symmetry. Then, we proceed to solve the classical equations of motion and to look for an appropriate symmetry as an input to the GAQ.

The Lagrangian for the MPT potential, with positive depth D and width $1/\alpha$, can be written as

$$L = \frac{1}{2}m\dot{x}^2 + \frac{D}{\cosh^2(\alpha x)} = \frac{1}{2}m\frac{\dot{\xi}^2}{1 + \alpha^2\xi^2} + \frac{D}{1 + \alpha^2\xi^2}, \quad (1)$$

where we have introduced the coordinate $\xi = \frac{\sinh(\alpha x)}{\alpha}$.

Let us solve the Euler–Lagrange equations for negative energy $E = -\epsilon$, $\epsilon > 0$. They are

$$\dot{\xi} = \sqrt{\frac{2}{m}[(1 + \alpha^2\xi^2)E + D]}, \quad (2)$$

i.e.

$$\frac{d\xi}{\sqrt{\frac{2\epsilon}{m}\sqrt{\left(\frac{D-\epsilon}{\epsilon}\right) - \alpha^2\xi^2}}} = dt; \quad (3)$$

the solution to which is

$$\xi = \sqrt{\frac{D-\epsilon}{\alpha^2\epsilon}} \sin\left(\sqrt{\frac{2\epsilon\alpha^2}{m}}t + \phi_0\right), \quad (4)$$

where $\phi_0 \equiv \sin^{-1} \frac{\alpha\xi_0}{\sqrt{\frac{D-\epsilon}{\epsilon}}}$ is the initial phase. Writing also the equation for the velocity, we arrive at a couple of solutions

$$\begin{aligned} \xi &= \xi_0 \cos\sqrt{\frac{2\epsilon\alpha^2}{m}}t + \sqrt{\frac{m}{2\epsilon\alpha^2}}\dot{\xi}_0 \sin\sqrt{\frac{2\epsilon\alpha^2}{m}}t, \\ \dot{\xi} &= \dot{\xi}_0 \cos\sqrt{\frac{2\epsilon\alpha^2}{m}}t - \sqrt{\frac{2\epsilon\alpha^2}{m}}\xi_0 \sin\sqrt{\frac{2\epsilon\alpha^2}{m}}t, \end{aligned} \quad (5)$$

where $\dot{\xi}_0 \equiv \sqrt{\frac{2\epsilon\alpha^2}{m}\sqrt{\frac{D-\epsilon}{\alpha^2\epsilon} - \xi_0^2}}$ is the initial velocity. They go to those of the harmonic oscillator in the limit in which $D \rightarrow \infty$, $\alpha \rightarrow 0$, but $\frac{2D\alpha^2}{m}$ is kept finite and equal to ω^2

(constant), that is

$$\xi = \xi_0 \cos \omega t + \frac{\dot{\xi}_0}{\omega} \sin \omega t, \quad \dot{\xi} = \dot{\xi}_0 \cos \omega t - \omega \xi_0 \sin \omega t. \quad (6)$$

Equations (5) behave as those of an harmonic oscillator with a frequency depending on the energy. In fact, the Hamiltonian can be written as

$$\begin{aligned} H &= \frac{1}{2} m \frac{\dot{\xi}^2}{1 + \alpha^2 \xi^2} - \frac{D}{1 + \alpha^2 \xi^2} = \frac{1}{2} m \frac{\dot{\xi}^2}{1 + \alpha^2 \xi^2} + \frac{D \alpha^2 \xi^2}{1 + \alpha^2 \xi^2} - D \\ &= \frac{1}{2} m \dot{\xi}^2 + \frac{1}{2} m \omega(\epsilon)^2 \xi^2 - D, \end{aligned} \quad (7)$$

where $\omega(\epsilon) \equiv \sqrt{\frac{2\epsilon\alpha^2}{m}}$, and this, up to the constant energy shift $-D$, is an harmonic oscillator with energy-dependent frequency $\omega(\epsilon)$.⁴ For positive energy they transform into the equations of motion for a ‘repulsive-like’ oscillator.

In order to write the Poisson bracket we observe the Poincaré–Cartan form $\Theta_{\text{PC}} = p \, dx - H \, dt = \frac{\partial L}{\partial \dot{x}} (dx - \dot{x} \, dt) + L \, dt$:

$$\begin{aligned} \Theta_{\text{PC}} &= \frac{m \dot{\xi}}{1 + \alpha^2 \xi^2} \, d\xi - \left(\frac{1}{2} m \frac{\dot{\xi}^2}{1 + \alpha^2 \xi^2} - \frac{D}{1 + \alpha^2 \xi^2} \right) dt \\ &= p_\xi \, d\xi - \left[(1 + \alpha^2 \xi^2) \frac{p_\xi^2}{2m} - \frac{D}{1 + \alpha^2 \xi^2} \right] dt, \end{aligned} \quad (8)$$

where the momentum canonically conjugate to ξ is

$$p_\xi \equiv \frac{\partial L}{\partial \dot{\xi}} = \frac{m \dot{\xi}}{1 + \alpha^2 \xi^2}. \quad (9)$$

A simple inspection of Θ_{PC} indicates that the basic Poisson bracket will acquire the canonical form

$$\{\xi, p_\xi\} = 1. \quad (10)$$

By examining the Poisson bracket of H with ξ and p_ξ , we observe that $\{H, \xi, (1 + \alpha^2 \xi^2) p_\xi\}$ ‘close’ a Lie subalgebra with structure constants depending on H , and that it is possible to close a true algebra by choosing an appropriate function of H to replace H . To be precise, the following classical functions close a $SO(2, 1)$ algebra

$$\left\langle \mathcal{E} \equiv 2\sqrt{D}\sqrt{H}, \mathcal{X} \equiv \frac{\sqrt{2}}{\sqrt{D}}\sqrt{H}\xi, \mathcal{P} \equiv \sqrt{2}(1 + \alpha^2 \xi^2)p_\xi \right\rangle \quad (11)$$

In fact, we find

$$\{\mathcal{E}, \mathcal{P}\} = -m\Omega^2 \mathcal{X}, \quad \{\mathcal{E}, \mathcal{X}\} = -\frac{1}{m}\mathcal{P}, \quad \{\mathcal{X}, \mathcal{P}\} = \frac{1}{D}\mathcal{E}, \quad (12)$$

where $\Omega \equiv \sqrt{\frac{2\alpha^2 D}{m}} = \omega(D)$, which is the frequency of the small oscillations (harmonic approximation).

For positive energy (scattering states) \mathcal{E} can be diagonalized in terms of real combinations of \mathcal{X} and \mathcal{P} , $\langle A \equiv \frac{1}{\Omega}\mathcal{E}, B \equiv \frac{1}{2\alpha}(\mathcal{P} + m\Omega\mathcal{X}), C \equiv \frac{1}{2\alpha}(\mathcal{P} - m\Omega\mathcal{X}) \rangle$, giving rise to the standard

⁴ See [21] for a unified derivation of different integrable potentials in one and two dimensions and a description of their solutions, among them the different versions of the Pöschl–Teller potentials. See also [22] for an interpretation of this system as an harmonic oscillator with position-dependent mass.

form of the $SL(2, \mathbb{R})$ algebra:

$$\{A, B\} = -B, \quad \{A, C\} = C, \quad \{B, C\} = A. \quad (13)$$

However, we are interested in describing bound states with negative energy. For these states, \mathcal{E} and \mathcal{X} are purely imaginary and, therefore, we must redefine $\mathcal{E}' \equiv -i\mathcal{E}$, $\mathcal{X}' \equiv -i\mathcal{X}$ and $\mathcal{P}' \equiv \mathcal{P}$. Then, the complex combinations ($L_0 \equiv \frac{1}{\Omega}\mathcal{E}'$, $L_- \equiv \frac{1}{2\alpha}(\mathcal{P}' - im\Omega\mathcal{X}')$, $L_+ \equiv \frac{1}{2\alpha}(\mathcal{P}' + im\Omega\mathcal{X}')$) satisfy the algebra

$$\{L_0, L_+\} = iL_+, \quad \{L_0, L_-\} = -iL_-, \quad \{L_+, L_-\} = iL_0 \quad (14)$$

which can be identified with the complex form of both $SU(2)$ and $SU(1, 1)$ algebras. Observing carefully the different algebras, it can be realized that $L_+ = B$, $L_- = C$ and $L_0 = -iA$. This means that the two diagonalizations are the same, the only difference being the use of A or L_0 , which are real for positive and negative energies, respectively. This allows us to consider the two cases simultaneously, with a single algebra, $SO(2, 1)$ or its different versions $SL(2, \mathbb{R})$ or $SU(1, 1)$, for both positive and negative energies, instead of using the $SU(2)$ algebra for describing bound states. The only difference will lie in the fact that for negative energies, some generators will be non-Hermitian, or the pair of creation–annihilation operators will not be the adjoint to each other; in other words, the representation obtained will fail to be unitary (this fact was already noted in [17]). We shall postpone the discussion of its implications to the final section.

3. Group approach to quantization

3.1. Brief report on the general theory

The group approach to quantization [8, 13, 23, 24] lies on the simple idea that essentially a quantum theory is nothing but a unitary irreducible representation of a Lie algebra usually, though not necessarily, associated with a Poisson subalgebra of the solution manifold of a classical system. The GAQ algorithm constitutes simply a technique for representing Lie groups in a geometric way using only canonical structures on a Lie group, the quantum states being complex functions on the group manifold itself. The carrier space supports the realization of all operators (and only those) in the enveloping algebra.

Let us remind the reader that on any Lie group with composition law $g'' = g' * g$, two different and compatible actions can be considered. In fact, the left and right actions

$$\begin{aligned} L_{g'} : G &\rightarrow G & g &\mapsto g'' \\ R_g : G &\rightarrow G & g' &\mapsto g'' \end{aligned}$$

are generated by right-invariant and left-invariant vector fields on the group G , \mathcal{X}^R and \mathcal{X}^L respectively, and both commute. This is a remarkable property which allows us to adopt one of those Lie algebras, let us say \mathcal{X}^R , as well as the associated enveloping algebra, as the set of physical operators $\hat{g}^i \sim X_{g^i}^R$, whereas the other is used to reduce the corresponding representation in a compatible way, by nullifying a maximal subalgebra (in the left enveloping algebra, in general), named *polarization*, on the (reduced) wavefunctions. Mostly, the relevant symmetry group is a central extension, \tilde{G} , of a Lie group G by $U(1)$. Aiming at representing the canonical Poisson bracket between x and p , the complex functions on \tilde{G} are then prompted to satisfy the $U(1)$ -constraint

$$\tilde{X}_\phi^R \Psi = i\Psi, \quad (15)$$

where \tilde{X}_ϕ^R is the central generator of \tilde{G} , associated with the phase invariance of wavefunctions.

The classical theory, including the space of coordinates, momenta and time, is recovered out of the group manifold in a manner similar to the way we obtain the solution manifold from the (q, p, t) space as the quotient by the kernel of the differential of the Poincaré–Cartan form Θ_{PC} . In fact, there is a generalized Poincaré–Cartan form on the group, the quantization 1-form Θ , such that $\mathcal{P} \equiv \tilde{G}/(\text{Ker } d\Theta \cap \text{Ker } \Theta)$ is a quantum manifold in the sense of Geometric Quantization [25], and $\mathcal{S} \equiv \tilde{G}/\text{Ker } d\Theta$ is the classical solution manifold. \mathcal{S} can be parametrized by functions of the form $\Theta(X_{g^i}^R)$, which are the Noether invariants.

3.2. The example of the relativistic harmonic oscillator

We resort to a rather non-trivial (1+1)-dimensional example to achieve two tasks. On one hand we exemplify the GAQ algorithm on a physical system, that is, a relativistic harmonic oscillator (RHO) or a particle moving on (1+1)-anti-de Sitter spacetime and, on the other, we arrive at precise results on the representations of $SU(1, 1) \approx SL(2, \mathbb{R})$ that will be required in the next section. Simpler examples can be found in [8].

Quantum symmetry differs from the classical counterpart in an extra phase (or $U(1)$) transformation which permits the realization of an exact invariance of action integrands (Lagrangians or Poincaré–Cartan forms), versus the semi-invariance achieved in classical mechanics. This is so even in the case of finite-dimensional semi-simple groups for which all central extensions are mathematically trivial. In fact, the actual central extension of the Lie algebra of such a symmetry points out to a specific co-adjoint orbit of the classical symmetry and, then, the phase space (solution manifold) of the classical system [26]. Let us comment very briefly on these details in relation to the case of the free (1+1)D non-relativistic and relativistic particle. The quantum symmetry of the Galilean particle obeys the following commutation relations:

$$[\tilde{X}_t^R, \tilde{X}_x^R] = 0, \quad [\tilde{X}_t^R, \tilde{X}_p^R] = -\frac{1}{m}\tilde{X}_x^R, \quad [\tilde{X}_x^R, \tilde{X}_p^R] = \tilde{X}_\phi^R, \quad (16)$$

which represent the classical Poisson algebra among x , p , $\frac{p^2}{2m}$ and 1, once the wavefunctions are constrained by the $U(1)$ -function condition (15). The algebra (16) constitutes a non-trivial central extension of that of the Galilei group by $U(1)$. In going to the relativistic case, the Poincaré group is also centrally extended, although trivially, in such a way that the corresponding algebra reads

$$[\tilde{X}_t^R, \tilde{X}_x^R] = 0, \quad [\tilde{X}_t^R, \tilde{X}_p^R] = -\frac{1}{m}\tilde{X}_x^R, \quad [\tilde{X}_x^R, \tilde{X}_p^R] = \frac{1}{mc^2}\tilde{X}_t^R + \tilde{X}_\phi^R. \quad (17)$$

In the non-relativistic limit, this algebra contracts to the non-trivial extension (16).

To describe the quantum $SL(2, \mathbb{R})$ symmetry, physically realized as a quantum relativistic harmonic oscillator⁵ [9–11], we can dilate (as the opposite to contract) the algebra (17) with an extra term in the rhs so that it contracts to the Poincaré algebra in the limit $\omega \rightarrow 0$ and to the non-relativistic harmonic oscillator, with angular frequency ω , in the $c \rightarrow \infty$ limit. We shall apply the group quantization mechanism to the resulting group parametrized with renamed time, position and momentum variables, (τ, y, π) , as well as the mass, μ , to prevent any confusion with analogous variables in the physical problem analysed in the next section. We then write

$$[\tilde{X}_\tau^R, \tilde{X}_y^R] = -\mu\omega^2\tilde{X}_\pi^R, \quad [\tilde{X}_\tau^R, \tilde{X}_\pi^R] = -\frac{1}{\mu}\tilde{X}_y^R, \quad [\tilde{X}_y^R, \tilde{X}_\pi^R] = \frac{1}{\mu c^2}\tilde{X}_\tau^R + \tilde{X}_\phi^R. \quad (18)$$

⁵ See also [27] for an alternative approach to the relativistic harmonic oscillator where the $SU(1, 1)$ group is also realized as a dynamical symmetry.

These commutation relations can be exponentiated to a group law in many (equivalent) ways, the next one being just a possibility:

$$\begin{aligned}\sin \omega \tau'' &= \frac{\omega}{\beta''} \left(\frac{\beta}{\mu c^2 \beta'} \pi' y' \sin \omega \tau' \sin \omega \tau + \frac{\beta \Pi'_0}{\mu \omega \beta' c} \cos \omega \tau' \sin \omega \tau \right. \\ &\quad \left. + \frac{\omega}{\beta' \mu c^3} y y' \Pi'_0 \sin \omega \tau' + \frac{\beta' \beta}{\omega} \cos \omega \tau \sin \omega \tau' + \frac{\pi' y}{\mu c^2 \beta'} \cos \omega \tau' \right) \\ y'' &= \frac{\pi' \beta}{\mu \omega} \sin \omega \tau + \beta y' \cos \omega \tau + \frac{y \Pi'_0}{\mu c} \\ \pi'' &= \frac{\omega y \pi}{\beta c^2} \left(\frac{\pi'}{\mu} \sin \omega \tau + \omega y' \cos \omega \tau \right) + \frac{\Pi_0}{c \beta} \left(\frac{\pi'}{\mu} \cos \omega \tau - \omega y' \sin \omega \tau \right) + \frac{\pi \Pi'_0}{\mu c} \\ \zeta'' &= \zeta' \zeta e^{\frac{i}{\hbar}(\delta'' - \delta' - \delta)},\end{aligned}\tag{19}$$

where

$$\begin{aligned}\Pi_0 &\equiv \sqrt{\mu c^2 + \pi^2 + \mu^2 \omega^2 y^2}, & \beta &\equiv \sqrt{1 + \frac{\omega^2 y^2}{c^2}} \\ \delta &\equiv -\mu c^2 \tau - f, & f &\equiv -\frac{2\mu c^2}{\omega} \tan^{-1} \left[\frac{\mu c^2}{\omega \pi y} (\beta - 1) \left(\frac{\Pi_0}{\mu c} - \beta \right) \right].\end{aligned}\tag{20}$$

From the group law above we derive directly the set of left-invariant vector fields, which are relevant in the reduction procedure, through a polarization algebra, and the generalization of the Poincaré–Cartan form

$$\begin{aligned}\tilde{X}_\tau^L &= \frac{\pi}{\mu} \frac{\partial}{\partial y} - \mu \omega^2 y \frac{\partial}{\partial \pi} + \frac{\Pi_0}{\mu c \beta^2} \frac{\partial}{\partial \tau}, & \tilde{X}_\pi^L &= \frac{\Pi_0}{\mu c} \frac{\partial}{\partial \pi} + \frac{m c y}{\Pi_0 + m c \hbar} \frac{1}{\Xi} \\ \tilde{X}_y^L &= \frac{\Pi_0}{\mu c} \frac{\partial}{\partial y} + \frac{\pi}{\mu c^2 \beta^2} \frac{\partial}{\partial \tau} - \frac{\mu c \pi}{\Pi_0 + \mu c \hbar} \frac{1}{\Xi}, & \tilde{X}_\phi^L &= \frac{\partial}{\partial \phi} \equiv \Xi\end{aligned}\tag{21}$$

as well as the right-invariant vector fields, which provide the quantum operators on $U(1)$ -complex functions on the group, once a polarization had been imposed. They are

$$\begin{aligned}\tilde{X}_\tau^R &= \frac{\partial}{\partial \tau}, \\ \tilde{X}_\pi^R &= \frac{\beta}{\mu \omega} \sin \omega \tau \frac{\partial}{\partial y} + \left(\frac{\omega y \pi}{\mu c^2 \beta} \sin \omega \tau + \frac{\Pi_0}{\mu c \beta} \cos \omega \tau \right) \frac{\partial}{\partial \pi} \\ &\quad + \frac{y}{\mu c^2 \beta} \cos \omega \tau \frac{\partial}{\partial \tau} - \frac{1}{(\Pi_0 + \mu c) \beta} \left(\Pi_0 y \cos \omega \tau - \frac{\pi c}{\omega} \sin \omega \tau \right) \frac{1}{\hbar} \Xi, \\ \tilde{X}_y^R &= \beta \cos \omega \tau \frac{\partial}{\partial y} + \left(\frac{\omega^2 y \pi}{c^2 \beta} \cos \omega \tau - \frac{\Pi_0 \omega}{c \beta} \sin \omega \tau \right) \frac{\partial}{\partial \pi} - \frac{y \omega}{c^2 \beta} \sin \omega \tau \frac{\partial}{\partial \tau} \\ &\quad + \frac{\mu}{(\Pi_0 + \mu c) \beta} (\Pi_0 \omega y \sin \omega \tau + \pi c \cos \omega \tau) \frac{1}{\hbar} \Xi.\end{aligned}\tag{22}$$

The structure of the algebra (18) prevents the existence of a first-order polarization subalgebra leading to the configuration ‘representation’ (there are first-order polarizations constituted by ladder operators leading to the Fock ‘representation’ [9, 10]). However, it is possible to look for a second-order polarization subalgebra of the left-enveloping algebra reproducing the configuration ‘representation’. The simplest choice is the algebra generated by

$$\left\langle \tilde{X}_\tau^{\text{HO}} \equiv (\tilde{X}_\tau^L)^2 - c^2 (\tilde{X}_y^L)^2 - \frac{2i\mu c^2}{\hbar} \tilde{X}_\tau^L + \frac{i\mu c^2 \omega}{\hbar} \Xi, \tilde{X}_\pi^L \right\rangle,\tag{23}$$

which must be imposed, along with the $U(1)$ -constraint, to complex functions $\Psi(\phi, y, \pi, \tau)$ on the extended group. The solutions are

$$\begin{aligned} \Xi\Psi = i\Psi &\rightarrow \Psi = e^{i\phi}\Phi(y, \pi, \tau) \\ \tilde{X}_\pi^L\Psi = 0 &\rightarrow \Psi = e^{i\phi} e^{\frac{i}{\hbar}f}\psi(y, \tau) \\ \tilde{X}_\tau^{\text{HO}}\Psi = 0 &\rightarrow \frac{1}{\beta^2} \frac{\partial^2\psi}{\partial\tau^2} - \frac{2i\mu c^2}{\hbar\beta^2} \frac{\partial\psi}{\partial\tau} - 2\omega^2 y \frac{\partial\psi}{\partial y} - c^2\beta^2 \frac{\partial^2\psi}{\partial y^2} - \frac{\mu^2 c^4}{\hbar^2\beta^2} \psi + \frac{\mu c^2\omega}{\hbar} \psi = 0, \end{aligned} \quad (24)$$

where f is the function that appears in (20). By restoring the rest-mass energy⁶ we get a Klein–Gordon-like equation from the third line in (24)

$$\hat{C}\varphi \equiv -\frac{c^2}{\omega^2} \square \varphi = N(N-1)\varphi, \quad (25)$$

where

$$\square \equiv \frac{1}{c^2\beta^2} \frac{\partial^2}{\partial\tau^2} - \frac{2\omega^2 y}{c^2} \frac{\partial}{\partial y} - \beta^2 \frac{\partial^2}{\partial y^2} \quad (26)$$

is the D'Alembertian in an anti-de Sitter spacetime and $N = \frac{\mu c^2}{\hbar\omega}$; see [10] where the connection to the motion in a homogeneous space under the group $SO(1, 2)$, that is, the anti-de Sitter universe, is studied. We use the notation $\hat{C}\varphi$ to highlight that the lhs in (25) is the quantum realization of the Casimir operator of the Lie algebra of $SL(2, \mathbb{R})$ [16].

Equation (25) can be solved in power series. Writing the energy wavefunctions in the form

$$\varphi_n \equiv e^{-ib_n\omega\tau} \beta^{-c_n} H_n^N, \quad (27)$$

and putting it in equation (25), we obtain the relations

$$b_n = c_n, \quad c_n = c_0 + n \equiv N + n, \quad (28)$$

as well as the differential equation for the polynomials H_n^N :

$$\left(1 + \frac{\zeta^2}{N}\right) \frac{d^2}{d\zeta^2} H_n^N - \frac{2}{N}(N+n-1)\zeta \frac{d}{d\zeta} H_n^N + \frac{n}{N}(2N+n-1)H_n^N = 0, \quad (29)$$

where $\zeta \equiv \sqrt{\frac{\mu\omega}{\hbar}} y$.

Equation (29) defines the so-called relativistic Hermite polynomials (RHP) originally found in [9] and further developed in [10]. There, we gave the corresponding Rodrigues' formula

$$H_n^N(\zeta) = (-1)^n \left(1 + \frac{\zeta^2}{N}\right)^{N+n} \frac{d^n}{d\zeta^n} \left[\left(1 + \frac{\zeta^2}{N}\right)^{-N} \right]. \quad (30)$$

The normalized solutions of equation (25), with respect to the scalar product

$$\langle \Psi, \Psi' \rangle = \int \Psi^*(y, \tau) \Psi'(y, \tau) dy d\tau, \quad (31)$$

which is invariant under the group $SL(2, \mathbb{R})$, are given by

$$\tilde{\Psi}_n^N(y, \tau) = C_n^N \Psi_n^N(y, \tau) \equiv C_n^N e^{-i(N+n)\omega\tau} \beta^{-(N+n)} H_n^N \left(\sqrt{\frac{\mu\omega}{\hbar}} y \right), \quad (32)$$

⁶ The actual way of centrally extending a Lie algebra with trivial cohomology, like that of the relativistic symmetry associated with the free particle or the harmonic oscillator, consists in redefining one particular generator, the energy in this case, with a term proportional to the central generator.

where

$$C_n^N = \sqrt{\frac{\omega}{2\pi}} \left(\frac{\mu\omega}{\hbar\pi}\right)^{1/4} \sqrt{\frac{N^{n-1/2}\Gamma(2N)\Gamma(N)}{n!\Gamma(2N+n)\Gamma(N-\frac{1}{2})}}. \tag{33}$$

Creation and annihilation operators for the RHO can be introduced simply by $\hat{Z} \equiv \frac{1}{\sqrt{2}}(\tilde{X}_y^R - i\mu\omega\tilde{X}_\pi^R)$ and $\hat{Z}^\dagger \equiv \frac{1}{\sqrt{2}}(-\tilde{X}_y^R - i\mu\omega\tilde{X}_\pi^R)$. They turn out to be, when acting on the solutions of equation (24) (see [10, 11]),

$$\begin{aligned} \hat{Z} &= \sqrt{\frac{\hbar}{2\mu\omega}} e^{i\omega t} \beta \left[\frac{\partial}{\partial y} + i \frac{\omega y}{c^2\beta^2} \frac{\partial}{\partial \tau} + \frac{\mu\omega y}{\hbar\beta^2} \right] \\ \hat{Z}^\dagger &= \sqrt{\frac{\hbar}{2\mu\omega}} e^{-i\omega t} \beta \left[-\frac{\partial}{\partial y} + i \frac{\omega y}{c^2\beta^2} \frac{\partial}{\partial \tau} + \frac{\mu\omega y}{\hbar\beta^2} \right]. \end{aligned} \tag{34}$$

These operators are adjoint to each other with respect to the scalar product (31). Their action on energy eigenstates Ψ_n^N is given by

$$\hat{Z}\Psi_n^N = n \frac{2N+n-1}{2N} \Psi_{n-1}^N, \quad \hat{Z}^\dagger\Psi_n^N = \Psi_{n+1}^N. \tag{35}$$

On normalized solutions (32) they become

$$\hat{Z}\bar{\Psi}_n^N = \sqrt{n \frac{2N+n-1}{2N}} \bar{\Psi}_{n-1}^N, \quad \hat{Z}^\dagger\bar{\Psi}_n^N = \sqrt{(n+1) \frac{2N+n}{2N}} \bar{\Psi}_{n+1}^N. \tag{36}$$

It should be remarked that the normalized solutions (32) are orthogonal on account of the integration in τ , but not in y . This causes problems in the time factorization in order to obtain the minimal (versus manifestly, or time-dependent) realization, in terms of just y (see [10] for a discussion and [11] for a detailed explanation), and a modification of the scalar product and the creation and annihilation operator is needed. In fact, the new scalar product is

$$\langle \Psi, \Psi' \rangle = \int \Psi^*(y)\Psi'(y) \frac{dy}{\beta^2}, \tag{37}$$

and the normalized solutions with respect to this scalar product, with the time dependence factorized out, are

$$\bar{\Psi}_n^{N'}(y) = C_n^{N'} \beta^{-(N+n)} H_n^N \left(\sqrt{\frac{\mu\omega}{\hbar}} y \right), \tag{38}$$

where

$$\begin{aligned} C_n^{N'} &= \sqrt{\frac{N+n}{N-\frac{1}{2}}} C_n^N = \sqrt{\frac{\omega}{2\pi}} \left(\frac{\mu\omega}{\hbar\pi}\right)^{1/4} \sqrt{\frac{N^{n-1}(N+n)\Gamma(2N)\Gamma(N+1)}{n!\Gamma(2N+n)\Gamma(N+\frac{1}{2})}} \\ &= \frac{1}{2\pi} \sqrt{\omega} \left(\frac{\mu\omega}{\hbar}\right)^{1/4} 2^N N^{n/2} \Gamma(N) \sqrt{\frac{N+n}{n!\Gamma(2N+n)}}. \end{aligned} \tag{39}$$

The modified creation and annihilation operators, adjoint to each other with respect to the new scalar product (37), are obtained [11] through the unitary transformation $\hat{Z}' = \hat{U} \hat{Z} \hat{U}^{-1}$, $\hat{Z}'^\dagger = \hat{U} \hat{Z}^\dagger \hat{U}^{-1}$, where \hat{U} is the unitary operator

$$\hat{U} = \sqrt{\frac{\hat{H}}{\hbar\omega(N - \frac{1}{2})}} e^{\frac{i}{\hbar}\tau(\hat{H} - \mu c^2)}. \quad (40)$$

Here \hat{H} is $i\hbar \frac{\partial}{\partial \tau}$ when acting on the manifestly covariant realization, and an infinite power expansion in $\frac{d}{dy}$ and y on the minimal realization (see [11]), which acquires the simple expression $\hbar\omega(N + n)$ on energy eigenfunctions (38). The expression of the new ladder operators in the minimal realization, acting on energy eigenfunctions (38), is [10]

$$\begin{aligned} \hat{Z}' &= \sqrt{\frac{\hbar}{2\mu\omega}} \sqrt{\frac{N+n-1}{N+n}} \beta \left[\frac{d}{dy} + \frac{\mu\omega y}{\hbar\beta^2} \frac{N+n}{N} \right] \\ \hat{Z}^\dagger &= \sqrt{\frac{\hbar}{2\mu\omega}} \sqrt{\frac{N+n+1}{N+n}} \beta \left[-\frac{d}{dy} + \frac{\mu\omega y}{\hbar\beta^2} \frac{N+n}{N} \right]. \end{aligned} \quad (41)$$

The RHP have been studied by different authors and related to other already known polynomials, such as Jacobi [28] or Gegenbauer [29] polynomials, and the essence of the latter is collected here since it is relevant for the next section. In fact, in [29] the following relation is proved:

$$H_n^N(v\sqrt{N}) = \frac{n!}{N^{\frac{n}{2}}} (1+v^2)^{\frac{n}{2}} C_n^N \left(\frac{v}{\sqrt{1+v^2}} \right), \quad (42)$$

where $C_n^N(v)$ are the Gegenbauer polynomials [30] directly related to the hypergeometric functions ${}_2F_1$. For negative index, $N \equiv -q$, we can also write

$$H_n^{-q}(\sqrt{q}v) \approx C_n^{q-n+\frac{1}{2}}(v). \quad (43)$$

It should be remarked that in [29] it is commented that ' $H_n^N(\xi)$ can actually be expressed directly as a (generalized) Gegenbauer polynomial in the form $C^{-N-n+\frac{1}{2}}(i\xi/\sqrt{N})$. This representation does not seem to be very useful, however'. We shall see in the next section that this relation will be rather useful in the description of the quantum MPT system.

4. The quantum Pöschl–Teller system

The commutation relations in (12) and (18) are formally analogous provided that we redefine in (18) the generator \tilde{X}_τ^R as $(\tilde{X}_\tau^R)' \equiv \tilde{X}_\tau^R + \mu c^2 \tilde{X}_\phi^R$, a redefinition which has been referred to as the restoring of the rest-mass energy and which, in mathematical terms, trivializes the central extension of the original $SO(2, 1)$ algebra⁷. We then aim at finding the quantum theory of the MPT dynamics in the quantum representation space of this symmetry and resorting to its enveloping algebra in search of the actual MPT Hamiltonian operator.

Let us proceed in a direct way, once the explicit computations have been developed for the $SO(2, 1)$ group in the example of the relativistic harmonic oscillator. First of all, we restore the standard notation t, x, p to represent time, coordinate and momentum for the MPT problem associated with a particle of mass m . The essential problem is now to find the explicit form of the operator $i\hbar \frac{\partial}{\partial t}$, the square of \mathcal{E} , acting on the wavefunctions representing the classical Poisson algebra (12) when rewritten in terms of the variable $x \equiv \sinh^{-1}(\alpha\xi)/\alpha$. To this end we rewrite (27) for a negative value $N \equiv -q < 0$ of the Bargmann index of the discrete series of the $SL(2, \mathbb{R})$ representations

$$\varphi_n^q \equiv e^{-ic_n\omega\tau} \left(1 + \frac{\omega^2}{c^2} y^2 \right)^{-\frac{q_n}{2}} H_n^{-q}, \quad (44)$$

⁷ The affine form in (18) is needed to perform the correct non-relativistic limit, which is a group contraction from $SO(2, 1)$ to the harmonic oscillator group.

Table 1. Expression of RHP with different values of the negative Bargmann index $-q$.

$q = 1$	$q = \frac{3}{2}$	$q = 1.8$
	$H_0^{-\frac{3}{2}}(x) = 1$	$H_0^{-1.8}(x) = 1$
$H_0^{-1}(x) = 1$	$H_1^{-\frac{3}{2}}(x) = 2x$	$H_1^{-1.8}(x) = 2x$
$H_1^{-1}(x) = 2x$	$H_2^{-\frac{3}{2}}(x) = -2 + \frac{8}{3}x^2$	$H_2^{-1.8}(x) = -2 + \frac{26}{9}x^2$
$H_2^{-1}(x) = 2(x^2 - 1)$	$H_3^{-\frac{3}{2}}(x) = -4x + \frac{16}{9}x^3$	$H_3^{-1.8}(x) = \frac{16}{81}x(-27 + 13x^2)$
$H_3^{-1}(x) = 0$	$H_4^{-\frac{3}{2}}(x) = 4$	$H_4^{-1.8}(x) = \frac{16}{243}(81 - 54x^2 + 13x^4)$
\vdots	$H_5^{-\frac{3}{2}}(x) = -\frac{40}{3}x$	$H_5^{-1.8}(x) = -\frac{32}{2187}x(405 - 90x^2 + 13x^4)$
$H_n^{-1}(x) = 0, n > 2$	\vdots	\vdots
	$H_{4+k}^{-\frac{3}{2}}(x)$ of degree k	$H_n^{-1.8}(x)$ of degree n

or, making explicit the \hbar constant, in terms of $u \equiv \frac{\zeta}{\sqrt{q}}, \zeta \equiv \sqrt{\frac{\mu\omega}{\hbar}}y$, and taking into account that $\frac{\omega^2}{c^2} = \frac{\mu\omega}{\hbar N}$, and $c_n = n - q$,

$$\varphi_n^q(\tau, u) = e^{-i(n-q)\omega\tau} \Psi_n^q(u) \equiv e^{-ic_n\omega\tau} (1 - u^2)^{\frac{q-n}{2}} H_n^{-q}(\sqrt{q}u). \tag{45}$$

In table 1 the expression of the RHP with different values of q are shown. They can also be derived from a Rodrigues’ formula analogous to (30) but substituting $-q$ for N .

The ladder operators for this system can also be obtained from the ones for the RHO given in the previous section, equation (34), simply by changing $N \rightarrow -q$ and performing the appropriate change of variables:

$$\begin{aligned} \hat{Z} &= \frac{1}{\sqrt{2q}} e^{i\omega\tau} \sqrt{1 - u^2} \left[\frac{\partial}{\partial u} - \frac{i}{\omega} \frac{u}{1 - u^2} \frac{\partial}{\partial \tau} \right] \\ \hat{Z}^\dagger &= \frac{1}{\sqrt{2q}} e^{-i\omega\tau} \sqrt{1 - u^2} \left[-\frac{\partial}{\partial u} - \frac{i}{\omega} \frac{u}{1 - u^2} \frac{\partial}{\partial \tau} \right], \end{aligned} \tag{46}$$

where the ‘rest-mass energy’ has been restored, in this case. The action of these operators on energy eigenstates has the simple form (similar to (35)):

$$\hat{Z}\varphi_n^q = n \frac{2q - n + 1}{2q} \varphi_{n-1}^q, \quad \hat{Z}^\dagger\varphi_n^q = \varphi_{n+1}^q. \tag{47}$$

From the action of the ladder operators, we conclude that the representation space has dimension $2q + 1$ (for q integer or half-integer), since $\hat{Z}\varphi_0^q = 0$ and $\hat{Z}^\dagger\varphi_{2q}^q = \varphi_{2q+1}^q$ is a ‘null vector’, i.e. a state that behaves as a vacuum state in the sense that $\hat{Z}\varphi_{2q+1}^q = 0$ (the concept of null state arose in the study of the representations of the Virasoro group [31–33] in the Verma-module approach). The irreducible finite-dimensional representation is obtained as the quotient by the ‘null vectors’ (see [18]). However, for q integer $\varphi_{2q+1}^q \equiv 0$ and the quotient process is trivial.

Therefore, unlike the RHO (discrete series representations), the MPT has only a finite number of (bound) states, realizing the irreducible (non-unitary) finite-dimensional representations of $SL(2, \mathbb{R})$ [17–19].

Let us try to derive the Schrödinger equation for the MPT potential from the Klein-Gordon equation of the relativistic harmonic oscillator (with negative Bargmann index). From equation (25) we can isolate the second ‘time’ derivative of φ :

$$\frac{\partial^2 \varphi}{\partial \tau^2} = -\omega^2(1-u^2) \left[-2u \frac{\partial \varphi}{\partial u} + (1-u^2) \frac{\partial^2 \varphi}{\partial u^2} + q(q+1)\varphi \right]. \quad (48)$$

Expressing the u -derivative in terms of x -derivative, from the relation $u \equiv \tanh(\alpha x)$,

$$\frac{\partial^2}{\partial x^2} = \alpha^2(1-u^2) \left[(1-u^2) \frac{\partial^2}{\partial u^2} - 2u \frac{\partial}{\partial u} \right], \quad (49)$$

and defining D through $q(q+1) \equiv N(N-1) = \frac{2mD}{\alpha^2 \hbar^2} = \left(\frac{2D}{\hbar\Omega}\right)^2$ we obtain

$$-\frac{\hbar^2 \alpha^2}{2m\omega^2} \frac{\partial^2}{\partial \tau^2} \varphi = \left[\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \varphi + \frac{D}{\cosh^2(\alpha x)} \varphi \right]. \quad (50)$$

This way, denoting $t \equiv \frac{2\omega}{\Omega} \tau$, $\hat{E} \equiv i\hbar \frac{\partial}{\partial t}$, and defining $\varphi \equiv e^{-\frac{i}{\hbar} \sqrt{D} \sqrt{-E} t} \chi$, ($-E \equiv \epsilon$), we arrive at the time-independent Schrödinger equation for a particle of mass m in a MPT potential with depth D and width $1/\alpha$:⁸

$$\frac{\hbar^2}{2m} \frac{d^2 \chi}{dx^2} + \left(E + \frac{D}{\cosh^2(\alpha x)} \right) \chi = 0. \quad (51)$$

The solutions to this equation were given in terms of Gegenbauer polynomials [7],

$$\chi_n^q(u) \approx (1-u^2)^{\frac{q-n}{2}} C_n^{q-n+\frac{1}{2}}(u), \quad (52)$$

which can now be compared with the time-independent part, $\Psi_n^q(u)$, of the functions of the relativistic harmonic oscillator (45) (through the relation (43)).

The scalar product for the minimal (time-independent) realization can be directly derived or obtained from that of the RHO, changing $\frac{dy}{\beta^2} \rightarrow \frac{du}{1-u^2}$:

$$\langle \Psi, \Psi' \rangle = \int_{-1}^1 \Psi^*(u) \Psi'(u) \frac{du}{1-u^2}. \quad (53)$$

The ladder operators for the time-independent realization can also be obtained from the ones for the RHO given in the previous section, simply by changing $N \rightarrow -q$ and performing the appropriate change of variables. On eigenfunctions they have the expression

$$\begin{aligned} \hat{Z}' &= \frac{1}{\sqrt{2q}} \sqrt{\frac{q-n+1}{q-n}} \sqrt{1-u^2} \left[\frac{d}{du} + \frac{u}{1-u^2} (q-n) \right] \\ \hat{Z}'^\dagger &= \frac{1}{\sqrt{2q}} \sqrt{\frac{q-n-1}{q-n}} \sqrt{1-u^2} \left[-\frac{d}{du} + \frac{u}{1-u^2} (q-n) \right], \end{aligned} \quad (54)$$

with the same action as in (47) on energy eigenstates Ψ_n^q .

These operators and their action coincide, up to a constant factor, with the ones given in [35].

The spectrum of the MPT Hamiltonian can also be derived from that of the RHO:

$$E_n = -\frac{\hbar^2 \alpha^2}{2m} (q-n)^2 = -\frac{\hbar^2 \Omega^2}{4D} (q-n)^2 = -\frac{D}{q(q+1)} (q-n)^2, \quad n = 0, 1, \dots, 2q. \quad (55)$$

⁸ See [34] for a similar derivation of this equation, although in a geometrical setting as generalization of the anti-de Sitter geometry, without using group-theoretical methods.

Let us look in detail at the obtained representation. We shall first consider the case of integer q . From equation (30) with $N = -q$, we observe that $H_n^{-q} = 0$ for $n > 2q$ (see table 1). Therefore, there are just $2q + 1$ states, in agreement with the previous statement that the representation is finite dimensional. Then, if q is an integer, all eigenvalues except one are doubly degenerated, the minimum being $E_0 = -\frac{\hbar^2 a^2}{2m} q^2 = -\frac{q}{q+1} D = E_{2q}$, and the maximum being $E_q = 0$. However, this degeneracy is only apparent, since the complete wavefunction for the states with the same energy, Ψ_n^q and Ψ_{2q-n}^q , $n = 0, \dots, q - 1$, are identical. Furthermore, if we consider the normalization of the states with the scalar product (53), it turns out that the state Ψ_q^q , the one with zero energy, is not normalizable. This means that the physical Hilbert space is spanned by Ψ_n^q , $n = 0, \dots, q - 1$, since the other states, Ψ_n^q , $n = q + 1, \dots, 2q$, are copies of them (we can also think of it as if they were not reachable by the action of creation operators, since the state Ψ_q^q is out of the Hilbert space).

If q is half-integer, from equation (30) with $N = -q$ (see table 1), we deduce that there are an infinite number of states. Their behaviour is as follows: for $n = 0, \dots, 2q$, H_n^q is a polynomial of degree n , as should be, but H_{2q+1}^q is a polynomial of degree zero, and then $H_{(2q+1)+k}^q$, $k = 1, 2, \dots$, is a polynomial of degree k . However, by the action of the ladder operators only the first $2q + 1$ states are reachable, and the representation is finite dimensional. Even more, the states Ψ_n^q and Ψ_{2q-n}^q , $n = 0, \dots, q - \frac{1}{2}$, are not identical and therefore there is a double degeneracy for all the states. However, if we take into account the normalizability with respect to the scalar product (53), it turns out that the physical Hilbert space is spanned by Ψ_n^q , $n = 0, \dots, q - 1/2$, the rest of states being not normalizable.

In summary, for the finite-dimensional (non-unitary) representations of $SL(2, \mathbb{R})$, from the $2q + 1$ states of the representation, Ψ_n^q , $n = 0, \dots, 2q$, only $[q] + 1$ are normalizable, Ψ_n^q , $n = 0, \dots, [q]$ (recall that $[q]$ was the greatest integer properly smaller than q), and these span the physical Hilbert space. These states are also orthogonal with respect to the scalar product (53), and the orthonormal basis is

$$\bar{\Psi}_n^q(u) = N_n^q \Psi_n^q(u), \quad N_n^q = \frac{2^{-q}}{\Gamma(q+1)} \sqrt{\frac{(q-n)\Gamma(2q-n+1)}{n!}}, \quad n = 0, \dots, [q]. \quad (56)$$

If we express the solutions in terms of Gegenbauer polynomials, the results are similar; the only difference is that for the non-normalizable states they are not defined. The reason is that the proportionality constant in (43) diverges for these cases.

5. Concluding remarks

The properties of the wavefunctions of the MPT potential are very different from that of $SU(2)$ representations, which also span finite-dimensional, though unitary, representations and, therefore, for j integer or half-integer, all $2j + 1$ states are orthogonal and normalizable. This clearly implies that we cannot use $SU(2)$ as the symmetry group for bound states. Furthermore, the use of $SU(2)$ leads to inconsistencies, since it predicts a double degeneracy in the eigenstates, something that is forbidden in one dimension. Despite this, it has been widely used in the literature; see, for instance, [5, 7].

An important fact of having finite-dimensional representations of $SL(2, \mathbb{R})$ instead of $SU(2)$ is that, going to the universal covering group of $SL(2, \mathbb{R})$, all positive values of q are allowed. In this case (see table 1) H_n^{-q} is a polynomial of degree n for all $n \in \mathbb{N}$, but taking

into account the normalizability with respect to the scalar product (53), only the first $[q] + 1$ states are normalizable, from $n = 0, 1, \dots, [q]$, and these span the physical Hilbert space⁹. Since $SU(2)$ is already simply connected, no real values other than integer or half-integer are allowed for the index j labelling its representations. This has relevant consequences from the physical point of view. Since $q(q+1) = \frac{2mD}{\alpha^2 \hbar^2} = \left(\frac{2D}{\hbar\Omega}\right)^2$, the restriction of q to integer and half-integer values (as happens for $SU(2)$ representations) leads to a formal quantization of the potential parameter D/α^2 (or rather $\frac{2D}{\hbar\Omega}$), whereas this does not happen for finite-dimensional $SL(2, \mathbb{R})$ representations, where all real values of q are allowed.

Probably, the most important reason to support the idea of describing the bound states of the MPT system by $SL(2, \mathbb{R})$ instead of $SU(2)$ is the harmonic limit, which consists in taking $D \rightarrow \infty, \alpha \rightarrow 0$ such that $\alpha^2 D$ is kept constant. Both the positive discrete series and the finite-dimensional representations of $SL(2, \mathbb{R})$ contract, under the limit $N \rightarrow \infty$ and $q \rightarrow \infty$, respectively, to the harmonic oscillator. In fact, from equation (30) it can be directly checked that $\lim_{N \rightarrow \pm\infty} H_n^N(x) = H_n(x)$. For the case of the finite-dimensional representations, the harmonic limit of the energies requires a previous redefinition, in such a way that $\lim_{D \rightarrow \infty} (E_n + D) = \hbar\Omega(\frac{1}{2} + n)$, that is, the spectrum of the harmonic oscillator with frequency $\Omega = \omega(D)$ is recovered. Even the ladder operators (46) go to the ladder operators of the harmonic oscillator (with frequency Ω) in the harmonic limit (see [35]). However, contracting the $SU(2)$ representations to that of the harmonic oscillator would require a negative spin index.

An important point to clarify is the physical interpretation, if any, of the non-normalizable states. For q integer, the only non-normalizable state is Ψ_q^q , which, under normalization (see (56)), turns out to be actually zero, $\bar{\Psi}_q^q = 0$. Even more, the action of the creation operator in (54) on Ψ_{q-1}^q is $\hat{Z}'^\dagger \Psi_{q-1}^q = 0$. Therefore, the irreducible subspace has dimension q , from $n = 0, 1, \dots, q-1$. A similar situation appears in the study of bound states for the square well potential. In this case, for appropriate values of the width and depth of the potential, there is a bound state with zero energy, but this state is either identically zero or non-normalizable (see [36] for a discussion), therefore it is unphysical and should be discarded.

For q half-integer, the situation is a little more involved, since the $(2q+1)$ -dimensional representation is obtained after a quotient-by-null-vector process, and in addition the normalization factor N_n^q in equation (56) is not valid for $n > q$, which should be zero since these states are non-normalizable. In this case, a normalization factor of the type $\theta(q-n)N_n^q$ should be appropriate, where $\theta(x)$ is a step function, i.e. it is zero for $x \leq 0$ and one for $x > 0$. A similar interpretation holds for q non-half-integer. In this case the representation is infinite dimensional, as can be seen from table 1 (in fact, it belongs to the discrete series of representations with negative Bargmann index, and it is non-unitary, see [17]). After a proper normalization with an step function $\theta(q-n)$, the physical Hilbert space is made up of only $[q] + 1$ normalizable states. On these states, the action of \hat{Z}' and \hat{Z}'^\dagger satisfies an equation analogous to (36) with $N = -q$.

Another question to tackle is the nature of the representations here obtained. For q integer or half-integer, we have obtained irreducible, non-unitary (true) representations of $SL(2, \mathbb{R})$, of dimension $2q+1$. Under normalization of the states (discarding the non-normalizable states) we have obtained a new kind of irreducible unitary representation which can be associated with $SL(2, \mathbb{R})$ but which cannot be, definitely, true representations of $SL(2, \mathbb{R})$. The creation and annihilation operators (54) close, together with the (square root of) the MPT Hamiltonian, an $SL(2, \mathbb{R})$ Lie algebra, although they lie in the enveloping algebra of the Lie algebra, since

⁹ This is in agreement with the WKB counting of bound states for a general potential [36], applied to the Pöschl–Teller potential, which turns out to be $N \approx \frac{1}{2} + \sqrt{q(q+1)}$, and this equals $q+1$ for large q .

they are non-local (they depend on the state). A detailed study of these representations is in progress.

This results can be extended to the Morse potential [4, 5, 37]. As in the present case there is a finite number of bound states, which are associated with a finite-dimensional, non-unitary representation of $SL(2, \mathbb{R})$ [38] (although in the literature they have also been associated with $SU(2)$).

As a last general comment, we should say that a more complete study of the Pöschl–Teller dynamics resorting to the GAQ of the $SL(2, \mathbb{C})$ group would be in order. In that case, the different parts of the spectrum would be more properly related to different (real) subgroups.

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